Calibrating word problems of groups
via the complexity of equivalence relations

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Three results connecting eqrels and f.g. groups

(1) There is a finitely presented group with a word problem which is a uniformly effectively inseparable equivalence relation.

(2) There is a finitely generated group of computable permutations with a word problem which is a universal co-computably enumerable equivalence relation.

(3) Each c.e. truth-table degree contains the word problem of a finitely generated group of computable permutations.

Main reference for this talk: eponymous 2018 paper by Nies and Sorbi in Math. Struct. in Comp. Science [8].

Definition of computable reducibility and universality

Definition

Given two equivalence relations R, S on $\mathbb{N}$, we say that R is computably reducible to S (notation: $R \leq S$) if there exists a computable function f such that, for every $x, y \in \mathbb{N}$,

$$x \; R \; y \Leftrightarrow f(x) \; S \; f(y).$$

Definition

Let $\mathcal{A}$ be a class of equivalence relations. An equivalence relation $R \in \mathcal{A}$ is called $\mathcal{A}$-universal if $S \leq R$ for every $S \in \mathcal{A}$.

Sample results on universal eqrels

- The isomorphism relation for various familiar classes of computable structures is Σ^1_1-universal: e.g. computable graphs (Fokina et al. 2012 [4]).
- 1-equivalence among c.e. sets is Σ^0_3-universal. (Fokina, Friedman and Nies 2012 [3]).
- Equality of functions $\Sigma^* \to \Sigma^*$ that are computable in quadratic time is a Π^0_1-universal equivalence relation. The functions are described by Turing programs. Ianovski et al. 2014 [5, Theorem 3.5].
- In contrast, Ianovski et al. show that there is no Π^0_n-universal equivalence relation for $n > 1$.
- In fact, for $n > 1$, each Π^0_n equivalence relation R there is a Δ^0_n relation S such that $S \not\leq R$.

In the talk, we will discuss four results that relate

Σ^0_1 universality and Π^0_1 universality
for equivalence relations

to

word problems and isomorphism problems
for finitely generated groups.

Σ^0_1-universal equivalence relations and

isomorphism of finitely presented groups

A little-known construction by C.F. Miller III

Write F_X for the free group on generators in X.

Some notation

- Given a group $G = \langle X; R\rangle = F_X/N$ where N is the normal closure of the set of relators R, the word problem is $\{(s,t)\colon st^{-1} \in N\}$.
- Write $=_G$ for this equivalence relation on F_X.

Theorem (C.F. Miller III, Group theoretic dec. problems, 1971[6])

(a) Given a Σ^0_1 eqrel E, one can effectively build a f.p. group $G_E = \langle X; R\rangle$, and a computable sequence of words $(w_i)_{i\in\mathbb{N}}$ in F_X such that $i\ E\ k \Leftrightarrow w_i =_G w_k$.

(b) Given a finite presentation $\langle X; R\rangle$ of a group G one can effectively find a computable family $(H^G_w)_{w\in F_X}$ of f.p. groups such that $v =_G w \Leftrightarrow H^G_v \cong H^G_w$ for all $v, w \in F_X$.

Finitely presented groups and Σ^0_1-universality

Corollary (to Miller's Theorem)

(i) There exists a f.p. group G such that $=_G$ is a Σ^0_1-universal eqrel.

(ii) The isomorphism relation $\cong_{f.p.}$ between finite presentations of groups is a Σ^0_1-universal eqrel.

Ianovski, Miller, Ng. and N. 2014 had asked (ii), not knowing that it had already been answered in the affirmative in [6].

Proof. Let E be a Σ^0_1-universal eqrel. Then

(i) by (a) of Miller's theorem, E is computably reducible to $=_{G_E}$, and thus $=_{G_E}$ is Σ^0_1-universal;

(ii) by (b) of Miller's theorem, $i\ E\ k \Leftrightarrow H^{G_E}_{w_i} \cong H^{G_E}_{w_k}$. This shows that E is computably reducible to $\cong_{f.p.}$. Hence $\cong_{f.p.}$ is Σ^0_1-universal.

A further question on $\cong_{\text{f.p.}}$ answered

- N. and Sorbi 2018 asked whether each pair of distinct equivalence classes of $\cong_{\text{f.p.}}$ is recursively inseparable.
- A negative answer was observed by Maurice Chiodo.

$G_{ab} = G/G'$ is the largest abelian quotient of a group G.

Observation by M. Chiodo, See Lyndon/Schupp, Logic Blog 2017, p. 18.

Let $\mathcal{A}$ be the set of finite presentations of groups G such that $G_{ab} \cong \mathbb{Z}$. This set $\mathcal{A}$ is recursive.

$\mathcal{A}$ contains all the presentations of $\mathbb{Z}$ and no presentation of $\mathbb{Z} \times \mathbb{Z}$. So these two equivalence classes can be separated by a recursive set.

A better version of the N. and Sorbi 2018 question

A group G is called perfect if $G' = G$. The finite presentations of perfect groups can be listed effectively. So there is a computable function P such that $P(n) = \langle X_n, R_n \rangle$ is a list of the finite presentations of perfect groups.

Let $E_P = \{\langle n, k \rangle : P(n) \cong P(k)\}$. Are any two equivalence classes of E_P recursively inseparable? If so, is E_P uniformly e.i.?

If Q is another such listing then E_P and E_Q are recursively isomorphic. So the answers don't depend on the choice of P. (Use a back and forth argument, together with the fact that $\cong_{\text{f.p.}}$ is Σ^0_1.)
If E_P is u.e.i. then it is already recursively isomorphic to $\sim_{PA}$.
This is because E_P has a "strong diagonal function", i.e.
a computable function g taking finite sets $D \subseteq \mathbb{N}$ as arguments such that $g(D) \notin [D]_E$ for each D.

Σ^0_1-universal equivalence relations and word problems

A f.p. group with u.e.i. word problem

Definition (See Soare 89, Exercise II.4.5)

Disjoint c.e. sets A, B are effectively inseparable if for each disjoint pair X, Y of c.e. sets, there is a 1-1 computable function f such that $f(X) \subseteq A$ and $f(Y) \subseteq B$. It suffices to ask this for the pair $X = \{e : \phi_e(e) = 0\}$, $Y = \{e : \phi_e(e) = 1\}$.

Theorem (First result in N. and Sorbi, 2018)

There is a finitely presented group H such that each pair of distinct equivalence classes of its word problem $=_H$ is effectively inseparable, uniformly in terms of elements of F_n representing the equivalence classes.

Note that the word problem of H is a Σ^0_1-universal eqrel by Andrews et al. 2014 [1].

Theorem (Recall, f.p. group with u.e.i. WP)

There is a finitely presented group H such that each pair of distinct equivalence classes of its word problem $=_D$ is effectively inseparable in a uniform way.

The proof has three main ingredients. (See the paper for detail.)

1. **Lemma.** Let $G = \langle X; R\rangle$ be a given f.p. group. Suppose $([1]_G, [w]_G)$ is e.i. where $w \in F_X$. Let $N = \mathtt{Ncl}_G(w)$.
Then, if $s, t \in N$ such that $s \neq_G t$, the pair $([s]_G, [t]_G)$ is e.i. uniformly in s, t.

2. A method of C.F. Miller builds a nontrivial f.p. group so that all its nontrivial quotients have an undecidable WP. This is done by encoding an e.i. pair into the word problem.

3. A construction from Lyndon/Schupp IV.3.5. embeds each countable group into a f.g. simple group.

Π^0_1-universal equivalence relations and word problems

F.g. subgroups of S_{rec}

Let α, β of two permutations on some set W. Then $\alpha\beta$ denotes the permutation such that $\alpha\beta(s) = \beta(\alpha(s))$ where $s \in W$.
Let S_{rec} denote the group of computable permutations of $\mathbb{N}$.

Fact

Suppose G is a f.g. subgroup of S_{rec}. Then the WP of G is Π^0_1.

Fact

Suppose that a f.g. group G has decidable WP. Then G is isomorphic to a subgroup of S_{rec}. (Use the right translation action of the generators.)

In contrast, Morozov 2000 [7] showed that there is a two-generator group with Π^0_1 word problem that is not embeddable into the group of computable permutations of $\mathbb{N}$.

Downwd closed classes of finitely generated groups

In the diagram below, arrows denote proper inclusions. All its classes of f.g. groups are closed under taking subgroups.

$$\begin{tikzcd}
\Sigma^0_1 \text{ word problem} & & \Pi^0_1 \text{ word problem} \\
& & \text{subgroup of } S_{\text{rec}} \arrow[u] \\
& \text{Computable WP} \arrow[uul] \arrow[ur] &
\end{tikzcd}$$

Automorphisms of negative numerations

Let $\nu\colon \mathbb{N} \to M$ be a numeration. Call a permutation ρ on M computable if there are computable $f, g\colon \mathbb{N} \to \mathbb{N}$ such that

$$\rho \circ \nu = \nu \circ f \text{ and } \rho^{-1} \circ \nu = \nu \circ g.$$

I.e., f "names" ρ and g "names" ρ^{-1} w.r.t. ν. These permutations form a group denoted $S_{\mathrm{rec}}(\nu)$.

FACT. If ν is a negative numeration (i.e. its kernel is Π^0_1) then each f.g. subgroup G of $S_{\mathrm{rec}}(\nu)$ has Π^0_1 word problem.

FACT. There is a single negative numeration ν such that each f.g. group with Π^0_1 WP occurs as a subgroup of $S_{\mathrm{rec}}(\nu)$.

To verify the second fact, one combines Morozov 2000 [7] (where the negative numeration depends on G) with the construction of a Π^0_1 universal eqrel in Ianovski et al. 2014 [5].

Theorem (N. and Sorbi 2018, second result)

There is a finitely generated group of computable permutations of $\mathbb{N}$ with word problem a Π^0_1-universal equivalence relation.

To prove this, let E be a Π^0_1-universal equivalence relation (Ianovski et al. [5]). By [5, Prop. 3.1] there is a computable function f such that

$$x \; E \; y \Leftrightarrow (\forall n)[f(x, n) = f(y, n)].$$

The construction of f shows that $f(x, n) \leq x$ for each x, n.

Basic setting for the proof

Fix a computable bijection $\langle \cdot\, , \cdot \rangle \colon \mathbb{Z} \times \mathbb{N} \to \mathbb{N}$. The domain of our computable permutations is a disjoint union of pairs of "columns"

$$C^i_x = \{2x + i\} \times \mathbb{N},$$

where $i = 0, 1$ and $x \in \mathbb{Z}$ for the rest of this proof.
The permutation σ shifts C^i_x to C^i_{x+1}:

$$\sigma(\langle 2x + i, n\rangle) = \langle 2x + 2 + i, n\rangle.$$

The permutation τ exchanges C^i_0 with C^{1-i}_0 and is the identity elsewhere:

$$\tau(\langle i, n\rangle) = \langle 1 - i, n\rangle \text{ and } \tau(\langle k, n\rangle) = \langle k, n\rangle \text{ if } k \neq 0, 1.$$

The permutation α encoding f

Recall that E is Π^0_1 universal eqrel, and f is computable, s.t.

$$x \; E \; y \Leftrightarrow (\forall n)[f(x, n) = f(y, n)].$$

The permutation α codes f in the sense that there exists a fixed computable sequence $(t_x)_{x\in\mathbb{N}}$ of terms in the free group generated by the symbols α, σ, τ, such that,
letting $G = \langle \alpha, \sigma, \tau\rangle \leq S_{\mathrm{rec}}$, for each $x, y \in \mathbb{N}$ we have

$$\forall n \, [f(x, n) = f(y, n)] \Leftrightarrow t_x =_G t_y. \tag{1}$$

For each x, n,

- α has a cycle of length $f(x, n) + 1$ in the interval $[n(x + 1), n(x + 1) + x]$ of C^0_x
- α is the identity on the remaining columns.

Defining terms $t_x(\alpha, \sigma, \tau)$

For $x \in \mathbb{N}$ we let $t_x = \sigma^x \alpha \sigma^{-x} \; \tau \; \sigma^x \alpha^{-1} \sigma^{-x}$.

- the permutation $t_x(\alpha, \sigma, \tau)$ only retains the encoding of the values $f(x, n)$, and erases all other information:
- it moves this information to the pair of columns C_0^0, C_0^1. In this way we can compare the values $f(x, n)$ and $f(y, n)$ applying t_x and t_y to α, σ, τ:

$$\forall n \, [f(x, n) = f(y, n)] \;\Leftrightarrow\; t_x = t_y.$$

In more detail, et α_x be the permutation given by $\alpha(\langle 2x, w \rangle) = \langle 2x, \alpha_x(w) \rangle$. We obtain

$$t_x(\langle u, w \rangle) = \begin{cases} \langle u, w \rangle, & \text{if } u \neq 0, 1, \\ \langle 1, \alpha_x(w) \rangle, & \text{if } u = 0, \\ \langle 0, (\alpha_x)^{-1}(w) \rangle, & \text{if } u = 1. \end{cases}$$

Background for the final result in N. and Sorbi 2018

For the rest of the talk, the "word problem" of a group $G = F_n/N$ is meant classically as the equivalence class of the identity element, i.e. N.

- Collins 1971 [2] showed that each r.e. truth table degree contains the word problem of a finitely presented group, extending the work of Fridman, Clapham, Boone and others showing this for c.e. Turing degrees.
- In contrast, Ziegler 1976 [9] constructed an r.e. bounded truth-table degree that does not contain the word problem of a finitely presented group.

Analog of Collins' result for Π_1^0 groups

Let us call a permutation σ fully primitive recursive if

both σ and σ^{-1} are primitive recursive.

The fully primitive recursive permutations form a group.

Theorem

Given an r.e. set S, there is a triple of fully primitive recursive permutations such that the group G generated by them has word problem truth table equivalent to S.

We prove this by modifying the construction of computable permutations α, σ, τ for our previous result.

Open questions

1. Is isomorphism of f.p. perfect groups an u.e.i. equivalence relation?

2. It there a f.g. group with u.e.i. word problem that also has a strong diagonal function? I.e., can the WP be recursively isomorphic to $\sim_{PA}$?

The third question connects Π_1^0 universality with a different area. It was asked by Ianovski, Miller, Ng and N. 2014 [5] and remains open to my knowledge.

3. Is isomorphism of finite-automata presentable equivalence relations Π_1^0-universal?

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